ON THE BRANDT λ^0-EXTENSIONS OF MONOIDS WITH ZERO

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ABSTRACT. We study algebraic properties of the Brandt λ^0-extensions of monoids with zero and non-trivial homomorphisms between the Brandt λ^0-extensions of monoids with zero. We introduce finite, compact topological Brandt λ^0-extensions of topological semigroups and countably compact topological Brandt λ^0-extensions of topological inverse semigroups in the class of topological inverse semigroups and establish the structure of such extensions and non-trivial continuous homomorphisms between such topological Brandt λ^0-extensions of topological monoids with zero. We also describe a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt λ^0-extensions of topological monoids with zeros.

1. Introduction and preliminaries

We shall follow the terminology of [1, 4, 15]. Given a semigroup S, we shall denote the set of idempotents of S by $E(S)$. A semigroup S with the adjoined unit [zero] will be denoted by S^1 [S^0] (cf. [4]). Next, we shall denote the unit (identity) and the zero of a semigroup S by 1_S and 0_S, respectively. Given a subset A of a semigroup S, we shall denote by $A^* = A \setminus \{0_S\}$ and $|A|$ = the cardinality of A. A semigroup S is called *regular* if for any $x \in S$ there exists $y \in S$ such that $xyx = x$, and it is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Such an element y is called *inverse* of x and it is denoted by x^{-1}. An inverse semigroup S is called *Clifford* if $xx^{-1} = x^{-1}x$, for all $x \in S$. We note that xx^{-1} is an idempotent in S for any $x \in S$, and that for any Clifford inverse semigroup S, every idempotent is in the center of S.

If $h\colon S \to T$ is a homomorphism (or a map) from a semigroup S into a semigroup T and if $s \in S$, then we denote the image of s under h by $(s)h$. A semigroup homomorphism $h\colon S \to T$ is called *trivial* if $(s)h = (t)h$ for all $s, t \in S$. A semigroup S is called *congruence-free* if it has only two congruences: the identical and the universal [15]. Obviously, a semigroup S is congruence-free if and only if every homomorphism h of S into an arbitrary semigroup T is an isomorphism "into" or is a trivial homomorphism.

Let S be a semigroup with zero and I_λ a set of cardinality $\lambda \geqslant 1$. We define the semigroup operation on the set $B_\lambda(S) = (I_\lambda \times S \times I_\lambda) \cup \{0\}$ as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

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and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in I_\lambda$ and $a, b \in S$. If $S = S^1$ then the semigroup $B_\lambda(S)$ is called the *Brandt λ-extension of the semigroup* S [9, 10]. Obviously, if S has zero then $\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) \mid 0_S \text{ is the zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B^0_\lambda(S) = B_\lambda(S)/\mathcal{J}$ and the semigroup $B^0_\lambda(S)$ is called the *Brandt λ^0-extension of the semigroup S with zero* [12].

Next, if $A \subseteq S$ then we shall denote $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ if A does not contain zero, and $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in I_\lambda$. If $\mathcal{I}$ is a trivial semigroup (i.e. $\mathcal{I}$ contains only one element), then we denote the semigroup $\mathcal{I}$ with the adjoined zero by $\mathcal{I}^0$. Obviously, for any $\lambda \geqslant 2$, the Brandt λ^0-extension of the semigroup $\mathcal{I}^0$ is isomorphic to the semigroup of $I_\lambda \times I_\lambda$-matrix units and any Brandt λ^0-extension of a semigroup with zero which also contains a non-zero idempotent contains the semigroup of $I_\lambda \times I_\lambda$-matrix units.

We shall denote the semigroup of $I_\lambda \times I_\lambda$-matrix units by B_λ and the subsemigroup of $I_\lambda \times I_\lambda$-matrix units of the Brandt λ^0-extension of a monoid S with zero by $B^0_\lambda(1)$. We always consider the Brandt λ^0-extension only of a monoid with zero. Obviously, for any monoid S with zero we have $B^0_1(S) = S$. Note that every Brandt λ-extension of a group G is isomorphic to the Brandt λ^0-extension of the group G^0 with adjoined zero. The Brandt λ^0-extension of the group with adjoined zero is called a *Brandt semigroup* [4, 15]. A semigroup S is a Brandt semigroup if and only if S is a completely 0-simple inverse semigroup [3, 14] (cf. also [15, Theorem II.3.5]). We also observe that the semigroup B_λ of $I_\lambda \times I_\lambda$-matrix units is isomorphic to the Brandt λ^0-extension of the two-element monoid with zero $S = \{1_S, 0_S\}$ and the trivial semigroup S (i. e. S is a singleton set) is isomorphic to the Brandt λ^0-extension of S for every cardinal $\lambda \geqslant 1$. We shall say that the Brandt λ^0-extension $B^0_\lambda(S)$ of a semigroup S is *finite* if the cardinal λ is finite.

In this paper we establish homomorphisms of the Brandt λ^0-extensions of monoids with zeros. We also describe a category whose objects are ingredients in the constructions of the Brandt λ^0-extensions of monoids with zeros. We introduce finite, compact topological Brandt λ^0-extensions of topological semigroups and countably compact topological Brandt λ^0-extensions of topological inverse semigroups in the class of topological inverse semigroups, and establish the structure of such extensions and non-trivial continuous homomorphisms between such topological Brandt λ^0-extensions of topological monoids with zero. We also describe a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt λ^0-extensions of topological monoids with zeros.

2. Some properties of the Brandt λ^0-extensions of semigroups

Gutik and Pavlyk [12] proved that for every cardinal $\lambda \geqslant 1$, the Brandt λ^0-extension of a semigroup S is a regular, orthodox, inverse, 0-simple or completely 0-simple semigroup if and only if such is also S. They also proved that for every cardinal $\lambda \geqslant 1$, the Brandt λ^0-extension of a semigroup S with zero is a congruence-free semigroup if and only if such is also S. The definition of the semigroup operation on the Brandt λ^0-extension of a semigroup implies the following:

Proposition 2.1. *Let $\lambda \geqslant 1$ be any cardinal. Then:*

(i) If T is a subsemigroup of a semigroup S then $B^0_\lambda(T)$ is a subsemigroup of $B^0_\lambda(S)$; and

(ii) If T is a left (resp., right, two-sided) ideal of a semigroup S then $B^0_\lambda(T)$ is a left (resp., right, two-sided) ideal in $B^0_\lambda(S)$.

Proposition 2.2. *Let $\lambda_1, \lambda_2 \geqslant 1$ be any cardinals, S a semigroup and S_1 the Brandt λ_1^0-extension of the semigroup S. Then the Brandt λ_2^0-extension of the semigroup S_1 is isomorphic to the Brandt λ^0-extension of the semigroup S for the cardinal $\lambda = \lambda_1 \cdot \lambda_2$.*

Proof. Let I_{λ_1} and I_{λ_2} be the sets of cardinality λ_1 and λ_2, respectively. We put $I_\lambda = I_{\lambda_1} \times I_{\lambda_2}$. Then the set I_λ has the cardinality $\lambda = \lambda_1 \cdot \lambda_2$. We define a map $h\colon B^0_{\lambda_2}(S_1) = B^0_{\lambda_2}(B^0_{\lambda_1}(S)) \to B^0_\lambda(S)$ as follows

$$(\alpha_2, (\alpha_1, s, \beta_1), \beta_2)h = ((\alpha_2, \alpha_1), s, (\beta_1, \beta_2)) \qquad \text{and} \qquad (0_2)h = 0,$$

for $s \in S$, $\alpha_1, \beta_1 \in I_{\lambda_1}$, $\alpha_2, \beta_2 \in I_{\lambda_2}$, and zeros 0_2 and 0 of semigroups $B^0_{\lambda_2}(B^0_{\lambda_1}(S))$ and $B^0_\lambda(S)$, respectively. Obviously, the map $h\colon B^0_{\lambda_2}(B^0_{\lambda_1}(S)) \to B^0_\lambda(S)$ is bijective and it preserves the semigroup operation, hence h is an isomorphism. □

The cardinal arithmetics and Proposition 2.2 imply the following corollaries:

Corollary 2.3. *Let S be a semigroup and λ_1, λ_2 any infinite cardinals. Then $B^0_{\lambda_2}(B^0_{\lambda_1}(S)) = B^0_\lambda(S)$, where $\lambda = \sup\{\lambda_1, \lambda_2\}$.*

Corollary 2.4. *$B^0_\lambda(B^0_\lambda(S)) = B^0_\lambda(S)$ for every infinite cardinal λ and any semigroup S.*

Corollary 2.5. *Let $\lambda, \nu \geqslant 1$ be any cardinals. Then the Brandt λ^0-extension of the semigroup of $I_\nu \times I_\nu$-matrix units is the semigroup of $I_{\lambda\cdot\nu} \times I_{\lambda\cdot\nu}$-matrix units.*

Corollary 2.6. *Let λ be any infinite cardinal. Then the Brandt λ^0-extension of the semigroup of $I_\lambda \times I_\lambda$-matrix units is the semigroup of $I_\lambda \times I_\lambda$-matrix units.*

Corollary 2.7. *Let $\lambda \geqslant 1$ be any cardinal. Then the Brandt λ^0-extension of a Brandt semigroup is a Brandt semigroup.*

Let S be a semigroup with zero 0_S and $\{S_\alpha\}_{\alpha\in\mathscr{A}}$ a family of subsemigroups of S such that $S = \bigcup_{\alpha\in\mathscr{A}} S_\alpha$ and $S_\alpha \cap S_\beta = S_\alpha \cdot S_\beta = 0_S$ for all distinct $\alpha, \beta \in \mathscr{A}$. Then the semigroup S is called an *orthogonal sum* of the semigroups $\{S_\alpha\}_{\alpha\in\mathscr{A}}$ and it is denoted by $\sum_{\alpha\in\mathscr{A}} S_\alpha$ (cf. [15]).

Proposition 2.8. *Let $\lambda \geqslant 1$ be any cardinal. Let a semigroup S be an orthogonal sum of a family of semigroups $\{S_\alpha\}_{\alpha\in\mathscr{A}}$ with zeros. Then the Brandt λ^0-extension $B^0_\lambda(S)$ of the semigroup S is isomorphic to the orthogonal sum $\sum_{\alpha\in\mathscr{A}} B^0_\lambda(S_\alpha)$.*

Proof. Obviously, $B^0_\lambda(S) = \bigcup_{\alpha\in\mathscr{A}} B^0_\lambda(S_\alpha)$ and $B^0_\lambda(S_\alpha) \cap B^0_\lambda(S_\beta) = \{0\}$ for all distinct $\alpha, \beta \in \mathscr{A}$, where 0 is the zero of the Brandt λ^0-extension $B^0_\lambda(S)$ of the semigroup S. Proposition 2.1 implies that $B^0_\lambda(S_\alpha)$ is a subsemigroup of $B^0_\lambda(S)$, for all $\alpha \in \mathscr{A}$. The semigroup operation in $B^0_\lambda(S)$ implies that for every distinct $\alpha, \beta \in \mathscr{A}$ and for any non-zero elements $(\gamma, s_\alpha, \delta) \in B^0_\lambda(S_\alpha)$ and $(\mu, t_\beta, \nu) \in B^0_\lambda(S_\beta)$ we have $(\gamma, s_\alpha, \delta) \cdot (\mu, t_\beta, \nu) = 0$. This completes the proof of the proposition. □

The semigroup operation on a semigroup S with $E(S) \neq \varnothing$ induces the *natural partial order* $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$, for $e, f \in E(S)$. If $E(S)$ has a zero then an idempotent $e \in (E(S))^*$ is called *primitive* if it is minimal in $(E(S))^*$ (cf. [15]).

An inverse semigroup S with zero is called *primitive inverse*, if every non-zero idempotent of S is primitive [15]. Since every primitive inverse semigroup is an orthogonal sum of Brandt semigroups (cf. Theorem II.4.3 of [15]), Proposition 2.8 and Corollary 2.7 imply:

Corollary 2.9. *Let $\lambda \geqslant 1$ be any cardinal. Then the Brandt λ^0-extension of a semigroup S is a primitive inverse semigroup if and only if such is also S.*

3. On homomorphisms of the Brandt λ^0-extensions of monoids with zero

The following proposition is obvious:

Proposition 3.1. *Let S and K be semigroups and let $\lambda \geqslant 2$. Then the homomorphism $h\colon B^0_\lambda(S) \to K$ is trivial if and only if its restriction $h|_{B^0_\lambda(1)}\colon B^0_\lambda(1) \to K$ is a trivial homomorphism.*

Proposition 3.2. *Let S be a monoid with zero, $\lambda \geqslant 1$ any cardinal, and $B^0_\lambda(S)$ the Brandt λ^0-extension of S. Then every non-trivial homomorphic image of $B^0_\lambda(S)$ is the Brandt λ^0-extension of some monoid with zero. Moreover, if T is the image of $B^0_\lambda(S)$ under a homomorphism h, then T is isomorphic to the Brandt λ^0-extension of the homomorphic image of the monoid $S_{\alpha,\alpha}$ under the homomorphism h for any $\alpha \in I_\lambda$.*

Proof. In the case $\lambda = 1$ the proof is trivial. Therefore we may assume that $\lambda \geqslant 2$. Let T be a semigroup and $h\colon B^0_\lambda(S) \to T$ a homomorphism. Without loss of generality we can assume that the homomorphism $h\colon B^0_\lambda(S) \to T$ is a surjective map. Note that $(0)h = 0_T$ is the zero in T, where 0 is the zero in $B^0_\lambda(S)$. By Theorem 1 of [8], the semigroup $B^0_\lambda(1)$ is congruence-free and thus Proposition 3.1 implies that the restriction $h|_{B^0_\lambda(1)}\colon B^0_\lambda(1) \to T$ of the homomorphism h is an isomorphism.

We fix $\alpha_0 \in I_\lambda$. Next we shall show that the semigroup T is the Brandt λ^0-extension of a semigroup T_0, where T_0 is the homomorphic image of the subsemigroup S_{α_0,α_0}, under the homomorphism h. For every $\alpha,\beta \in I_\lambda$, we denote $1_{\alpha,\beta} = ((\alpha, 1_S, \beta))h$ and $T^*_{\alpha,\beta} = \{((\alpha, s, \beta))h \mid s \in S\} \setminus \{0_T\}$. First we show that for any $\alpha,\beta,\gamma,\delta \in I_\lambda$ we have $\left|T^*_{\alpha,\beta}\right| = \left|T^*_{\gamma,\delta}\right|$.

We define the maps $\varphi^{(\gamma,\delta)}_{(\alpha,\beta)}\colon T^*_{\alpha,\beta} \to T^*_{\gamma,\delta}$ and $\varphi^{(\alpha,\beta)}_{(\gamma,\delta)}\colon T^*_{\gamma,\delta} \to T^*_{\alpha,\beta}$ by the formulae $(x)\varphi^{(\gamma,\delta)}_{(\alpha,\beta)} = 1_{\gamma,\alpha} \cdot x \cdot 1_{\beta,\delta}$ and $(x)\varphi^{(\alpha,\beta)}_{(\gamma,\delta)} = 1_{\alpha,\gamma} \cdot x \cdot 1_{\delta,\beta}$. Then for any $s_{\alpha,\beta} = ((\alpha, s, \beta))h \in T^*_{\alpha,\beta}$, $s \in S \setminus \{0\}$, we get

$$
\begin{aligned}
(s_{\alpha,\beta})\big(\varphi^{(\gamma,\delta)}_{(\alpha,\beta)} \circ \varphi^{(\alpha,\beta)}_{(\gamma,\delta)}\big) &= 1_{\alpha,\gamma} \cdot 1_{\gamma,\alpha} \cdot s_{\alpha,\beta} \cdot 1_{\beta,\delta} \cdot 1_{\delta,\beta} = \\
&= ((\alpha, 1_s, \gamma))h \cdot ((\gamma, 1_s, \alpha))h \cdot ((\alpha, s, \beta))h \cdot ((\beta, 1_s, \delta))h \cdot ((\delta, 1_s, \beta))h = \\
&= \big((\alpha, 1_s, \gamma) \cdot (\gamma, 1_s, \alpha) \cdot (\alpha, s, \beta) \cdot (\beta, 1_s, \delta) \cdot (\delta, 1_s, \beta)\big)h = \\
&= ((\alpha, s, \beta))h = s_{\alpha,\beta},
\end{aligned}
$$

and similarly

$$
(s_{(\gamma,\delta)})\big(\varphi^{(\alpha,\beta)}_{(\gamma,\delta)} \circ \varphi^{(\gamma,\delta)}_{(\alpha,\beta)}\big) = s_{\gamma,\delta}.
$$

Hence the compositions $\varphi^{(\gamma,\delta)}_{(\alpha,\beta)} \circ \varphi^{(\alpha,\beta)}_{(\gamma,\delta)}\colon T^*_{\alpha,\beta} \to T^*_{\alpha,\beta}$ and $\varphi^{(\alpha,\beta)}_{(\gamma,\delta)} \circ \varphi^{(\gamma,\delta)}_{(\alpha,\beta)}\colon T^*_{\gamma,\delta} \to T^*_{\gamma,\delta}$ are the identity maps. Therefore the maps $\varphi^{(\gamma,\delta)}_{(\alpha,\beta)}$ and $\varphi^{(\alpha,\beta)}_{(\gamma,\delta)}$ are mutually invertible and hence we have that $\left|T^*_{\alpha,\beta}\right| = \left|T^*_{\gamma,\delta}\right| = |T_0 \setminus \{0_T\}|$. This implies that $T = I_\lambda \times (T_0 \setminus \{0_T\}) \times I_\lambda \cup \{0_T\}$.

Elementary calculations shows that for all $s, t \in S \setminus \{0_S\}$ we have

(1) $s_{\alpha,\beta} \cdot t_{\beta,\gamma} = \begin{cases} (st)_{\alpha,\gamma}, & \text{if } st \neq 0_S; \\ 0_T, & \text{if } st = 0_S; \end{cases}$
(2) $s_{\alpha,\beta} \cdot t_{\gamma,\delta} = 0_T$ for $\beta \neq \gamma$; and
(3) $s_{\alpha,\beta} \cdot 0_T = 0_T \cdot s_{\alpha,\beta} = 0_T$,

$\alpha, \beta, \gamma, \delta \in I_\lambda$, and hence T is the Brandt λ^0-extension of the semigroup T_0. This proves the last assertion of the proposition. □

Since a homomorphic image of a subgroup is a subgroup, Propositions 3.1 and 3.2 imply the following:

Corollary 3.3. *Every non-trivial homomorphic image of a Brandt semigroup is a Brandt semigroup.*

Proposition 3.4. *Let S and T be monoids with zeros, and let λ_1 and λ_2 be any cardinals such that $\lambda_2 \geqslant \lambda_1 \geqslant 1$. Let $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ be a non-trivial homomorphism. Suppose that the monoid T has the following properties:*

1) *T does not contain the semigroup of $I_{\lambda_1} \times I_{\lambda_1}$-matrix units; and*
2) *T does not contain the semigroup of 2×2-matrix units B_2 such that the zero of B_2 is the zero of T.*

Then the following assertions hold:

(*i*) *The image of zero 0_S of the semigroup $B^0_{\lambda_1}(S)$ under the homomorphism σ is the zero of the semigroup $B^0_{\lambda_2}(T)$;*
(*ii*) *If $(\alpha, 1_S, \beta)$ and $(\gamma, 1_S, \delta)$ are distinct elements of the Brandt λ_1-extension of the semigroup S, $\alpha, \beta, \gamma, \delta \in I_{\lambda_1}$, such that $((\alpha, 1_S, \beta))\sigma \in T_{\mu,\nu}$ and $((\gamma, 1_S, \delta))\sigma \in T_{\iota,\kappa}$ for some $\mu, \nu, \iota, \kappa \in I_{\lambda_2}$, then $T^*_{\mu,\nu} \cap T^*_{\iota,\kappa} = \varnothing$.*

Proof. Suppose to the contrary, that $(0_S)\sigma$ is not the zero of the semigroup $B^0_{\lambda_2}(T)$. Since the element $(0_S)\sigma$ is an idempotent of $B^0_{\lambda_2}(T)$, there exists $\alpha \in I_{\lambda_2}$ such that $(0_S)\sigma \in T^*_{\alpha,\alpha}$. Since $B^0_{\lambda_1}(1)$ is a congruence-free semigroup, σ is a non-trivial homomorphism and T does not contain the semigroup of $I_{\lambda_1} \times I_{\lambda_1}$-matrix units. We can conclude that there exist $\gamma, \delta \in I_{\lambda_2}$ such that $\gamma \neq \alpha$ or $\delta \neq \alpha$ and $(B^0_{\lambda_1}(S))\sigma \cap T^*_{\gamma,\delta} \neq \varnothing$. Let $x \in (B^0_{\lambda_1}(S))\sigma \cap T^*_{\gamma,\delta}$. If $\gamma \neq \alpha$ then the element $(0_S)\sigma \cdot x$ is the zero of $B^0_{\lambda_2}(T)$ and if $\delta \neq \alpha$ then the element $x \cdot (0_S)\sigma$ is the zero of $B^0_{\lambda_2}(T)$. But $x = (S)\sigma$ for some non-zero element s of the semigroup $B^0_{\lambda_1}(1)$. Therefore

$$x \cdot (0_S)\sigma = (s \cdot 0_S)\sigma = (0_S)\sigma \qquad \text{and} \qquad (0_S)\sigma \cdot x = (0_S \cdot s)\sigma = (0_S)\sigma,$$

a contradiction. Hence the statement (*i*) holds.

Next we shall show that there does not exist $\alpha_0 \in I_{\lambda_2}$ such that $((\alpha, 1_S, \beta))\sigma$, $((\beta, 1_S, \alpha))\sigma \in T^*_{\alpha_0,\alpha_0}$ for distinct $\alpha, \beta \in I_{\lambda_1}$. Suppose to the contrary. Then since T_{α_0,α_0} is a subsemigroup in $B^0_{\lambda_2}(T)$ and σ is a non-trivial homomorphism Proposition 3.1 implies

$$((\alpha, 1_S, \beta)))\sigma \cdot ((\beta, 1_S, \alpha))\sigma = ((\alpha, 1_S, \beta) \cdot (\beta, 1_S, \alpha))\sigma = ((\alpha, 1_S, \alpha))\sigma \in T^*_{\alpha_0,\alpha_0}$$

and

$$((\beta, 1_S, \alpha))\sigma \cdot ((\alpha, 1_S, \beta)))\sigma = ((\beta, 1_S, \alpha) \cdot (\alpha, 1_S, \beta))\sigma = ((\beta, 1_S, \beta))\sigma \in T^*_{\alpha_0,\alpha_0},$$

and hence

$$(0_S)\sigma = ((\alpha, 1_S, \alpha) \cdot (\beta, 1_S, \beta))\sigma = ((\alpha, 1_S, \alpha))\sigma \cdot ((\beta, 1_S, \beta))\sigma \in T_{\alpha_0,\alpha_0}.$$

Then by statement (i), the element $(0_S)\sigma$ is the zero of the semigroup $B^0_{\lambda_2}(T)$. This contradicts the assumption that the monoid T does not contain the semigroup of 2×2-matrix units B_2 such that the zero of B_2 is the zero of T.

In the next step we shall show that there does not exist $\alpha_0 \in I_{\lambda_2}$ such that $((\alpha, 1_S, \beta))\sigma \in T^*_{\alpha_0,\alpha_0}$. Otherwise we would have

$$(\alpha, 1_S, \beta)\cdot(\beta.1_S, \alpha) = (\alpha, 1_S, \alpha),$$

and since the homomorphism σ is non-trivial, we would have

$$((\alpha, 1_S, \beta))\sigma \cdot ((\beta.1_S, \alpha))\sigma = ((\alpha, 1_S, \alpha))\sigma \in T^*_{\alpha_0,\alpha_0},$$

and hence $((\alpha, 1_S, \alpha))\sigma \in T^*_{\alpha_0,\alpha_0}$. Therefore $((\alpha, 1_S, \beta))\sigma \in T^*_{\alpha_0,\alpha_0}$ and $((\beta.1_S, \alpha))\sigma \in T^*_{\alpha_0,\alpha_0}$. This contradicts the previous statement.

We shall show that there does not exist two distinct non-zero idempotents $(\alpha, 1_S, \alpha)$ and $(\beta, 1_S, \beta)$ in $B^0_{\lambda_1}(S)$, $\alpha, \beta \in I_{\lambda_1}$, such that $((\alpha, 1_S, \alpha))\sigma, ((\beta, 1_S, \beta))\sigma \in T^*_{\alpha_0,\alpha_0}$ for some $\alpha_0 \in I_{\lambda_2}$. Suppose to the contrary. Then

$$(\alpha, 1_S, \alpha) = (\alpha, 1_S, \beta)\cdot(\beta, 1_S, \alpha) \qquad \text{and} \qquad (\beta, 1_S, \beta) = (\beta, 1_s, \alpha)\cdot(\alpha, 1_S, \beta),$$

and hence

$$((\alpha, 1_S, \alpha))\sigma = ((\alpha, 1_S, \beta))\sigma\cdot((\beta, 1_S, \alpha))\sigma \quad \text{and} \quad ((\beta, 1_S, \beta))\sigma = ((\beta, 1_s, \alpha))\sigma\cdot((\alpha, 1_S, \beta))\sigma.$$

Since σ is a non-trivial homomorphism, Proposition 3.1 implies that $(\alpha, 1_S, \beta)\sigma \in T^*_{\alpha_0,\alpha_0}$ and $(\beta, 1_s, \alpha)\sigma \in T^*_{\alpha_0,\alpha_0}$. Hence $(\alpha, 1_S, \alpha)\sigma \in T^*_{\alpha_0,\alpha_0}$ and $(\beta, 1_s, \beta)\sigma \in T^*_{\alpha_0,\alpha_0}$. This is in contradiction with the previous statement.

In order to complete our proof we need to prove that there do not exist $\mu, \nu \in I_{\lambda_2}$ such that $((\alpha, 1_S, \beta))\sigma$, $((\gamma, 1_S, \delta))\sigma \in T^*_{\mu,\nu}$ for distinct non-idempotent elements $(\alpha, 1_S, \beta)$ and $(\gamma, 1_S, \delta)$ from the semigroup $B_{\lambda_1}(S)$. Suppose to the contrary. We consider only the case $\alpha \neq \gamma$. In the case $\beta \neq \delta$ the proof is similar. Then since σ is non-trivial homomorphism, Proposition 3.1 implies

$$((\alpha, 1_S, \alpha))\sigma = ((\alpha, 1_S, \beta)\cdot(\beta, 1_S, \alpha))\sigma = ((\alpha, 1_S, \beta))\sigma\cdot((\beta, 1_S, \alpha))\sigma \in T^*_{\mu,\nu}$$

and

$$((\gamma, 1_S, \gamma))\sigma = ((\gamma, 1_S, \delta)\cdot(\delta, 1_S, \gamma))\sigma = ((\gamma, 1_S, \delta))\sigma\cdot((\delta, 1_S, \gamma))\sigma \in T^*_{\mu,\nu}.$$

But this contradicts the previous statement. The obtained contradiction implies the statement of the proposition. $\square$

The following example shows that Proposition 3.4 fails in the case when the semigroup T contains the semigroup of 2×2-matrix units B_2 such that zero of T is zero of B_2.

Example 3.5. Let B_2 be the semigroup of 2×2-matrix units. Let S be the semigroup B_2 with the adjoined identity and $I_4 = \{1, 2, 3, 4\}$. We define a map

$h\colon B_4 \to B_4(S)$ as follows:

$$
\begin{array}{ll}
(0)h = 0, & \\
((1,1))h = (1,(1,1),1), & ((2,2))h = (1,(2,2),1), \\
((3,3))h = (2,(1,1),2), & ((4,4))h = (2,(2,2),2), \\
((1,2))h = (1,(1,2),1), & ((2,1))h = (1,(2,1),1), \\
((1,3))h = (1,(1,1),2), & ((3,1))h = (2,(1,1),1), \\
((1,4))h = (1,(1,2),2), & ((4,1))h = (2,(2,1),1), \\
((2,3))h = (1,(2,1),2), & ((3,2))h = (2,(1,2),1), \\
((2,4))h = (1,(2,2),2), & ((4,2))h = (2,(2,2),1), \\
((3,4))h = (2,(1,2),2), & ((4,3))h = (2,(2,1),2),
\end{array}
$$

where by 0 we denote the zeros of semigroups B_4 and $B_4(S)$. Elementary calculation shows that the map $h\colon B_4 \to B_4(S)$ is a semigroup homomorphism.

The following example shows that Proposition 3.4 fails in the case when the semigroup T contains the semigroup of $I_{\lambda_1} \times I_{\lambda_1}$-matrix units B_{λ_1}.

Example 3.6. Let λ_1 and λ_2 be any cardinals $\geqslant 2$. Let P be the semigroup of $I_{\lambda_1} \times I_{\lambda_1}$-matrix units B_{λ_1} with the adjoined identity ι_1, 0_1 be the zero of B_{λ_1} and let $z \notin P$. We extend the semigroup operation onto $T = P \cup \{z\}$ as follows:

$$s \cdot z = z \cdot s = z \cdot z = z, \qquad \text{for all } s \in P.$$

Obviously, z is the zero of the semigroup T.

Let S be a monoid with the zero 0_S of cardinality $\geqslant 3$. We define a map $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ as follows: fix arbitrary $\alpha \in I_{\lambda_2}$ and put

$$(x)\sigma = \begin{cases} (\alpha,(\beta,\iota_1,\gamma),\alpha), & \text{if } x = (\beta,s,\gamma) \text{ is a non-zero element of } B^0_{\lambda_1}(S); \\ (\alpha,0_1,\alpha), & \text{if } x \text{ is zero of } B^0_{\lambda_1}(S). \end{cases}$$

Obviously, such a map $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ is a semigroup homomorphisms.

Definition 3.7. Let λ be any cardinal $\geqslant 2$. We shall say that a semigroup S has the *$\mathcal{B}^*$-property* if S does not contain the semigroup of 2×2-matrix units and that S has the *$\mathcal{B}^*_\lambda$-property* if S satisfies the following conditions:

1) T does not contain the semigroup of $I_\lambda \times I_\lambda$-matrix units; and
2) T does not contain the semigroup of 2×2-matrix units B_2 such that the zero of B_2 is the zero of T.

Obviously, a semigroup S has the $\mathcal{B}^*$-property if and only if S has the $\mathcal{B}^*_2$-property, and hence Proposition 3.4 implies:

Corollary 3.8. *Let S and T be monoids with zeros, and λ_1 and λ_2 any cardinals such that $\lambda_2 \geqslant \lambda_1 \geqslant 1$. Let $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ be a non-trivial homomorphism. If the monoid T has the $\mathcal{B}^*$-property, then the following assertions hold:*

(i) The image of the zero 0_S of the semigroup $B^0_{\lambda_1}(S)$ under the homomorphism σ is the zero of the semigroup $B^0_{\lambda_2}(T)$; and

(ii) If $(\alpha,1_S,\beta)$ and $(\gamma,1_S,\delta)$ are distinct elements of the Brandt λ_1-extension of the semigroup S, $\alpha,\beta,\gamma,\delta \in I_{\lambda_1}$, such that $((\alpha,1_S,\beta))\sigma \in T_{\mu,\nu}$ and $((\gamma,1_S,\delta))\sigma \in T_{\iota,\kappa}$ for some $\mu,\nu,\iota,\kappa \in I_{\lambda_2}$, then $T^*_{\mu,\nu} \cap T^*_{\iota,\kappa} = \varnothing$.

Corollary 3.9. *Let S and T be monoids with zeros, and λ_1 and λ_2 any cardinals such that $\lambda_2 \geqslant \lambda_1 \geqslant 1$. Let $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ be a non-trivial homomorphism. Let $\alpha, \beta \in I_{\lambda_1}$ and $(\alpha, 1_S, \beta)\sigma \in T^*_{\gamma,\delta}$. Suppose that the monoid T has the $\mathcal{B}^*$-property. Then the following assertions hold:*

(i) *If $(\alpha, s, \beta)\sigma$ is a non-zero element of the semigroup $B^0_{\lambda_2}(T)$, then $(\alpha, s, \beta)\sigma \in T^*_{\gamma,\delta}$; and*
(ii) *If $(\alpha, s, \beta)\sigma$ is a non-zero element of the semigroup $B^0_{\lambda_2}(T)$, then such is also $(\alpha_1, s, \beta_1)\sigma$ for all $\alpha_1, \beta_1 \in I_{\lambda_1}$.*

Proof. The statement (i) follows from Proposition 2.1(ii).

Suppose there exist $\alpha_1, \beta_1 \in I_{\lambda_1}$ such that $(\alpha_1, s, \beta_1)\sigma = 0_2$ is the zero of the semigroup $B^0_{\lambda_2}(T)$. Then

$$\begin{aligned}
(\alpha, s, \beta)\sigma &= ((\alpha, 1_S, \alpha_1) \cdot (\alpha_1, s, \beta_1) \cdot (\beta_1, 1_S, \beta))\sigma = \\
&= ((\alpha, 1_S, \alpha_1))\sigma \cdot ((\alpha_1, s, \beta_1))\sigma \cdot ((\beta_1, 1_S, \beta))\sigma = \\
&= ((\alpha, 1_S, \alpha_1))\sigma \cdot 0_2 \cdot ((\beta_1, 1_S, \beta))\sigma = 0_2.
\end{aligned}$$

The obtained contradiction implies the assertion (ii) of the corollary. □

The following theorem describes all non-trivial homomorphisms of the Brandt λ^0-extensions of monoids with zeros.

Theorem 3.10. *Let λ_1 and λ_2 be cardinals such that $\lambda_2 \geqslant \lambda_1 \geqslant 1$. Let $B^0_{\lambda_1}(S)$ and $B^0_{\lambda_2}(T)$ be the Brandt λ^0_1- and λ^0_2-extensions of monoids S and T with zero, respectively. Let $h\colon S \to T$ be a homomorphism such that $(0_S)h = 0_T$ and suppose that $\varphi\colon I_{\lambda_1} \to I_{\lambda_2}$ is a one-to-one map. Let e be a non-zero idempotent of T, H_e a maximal subgroup of T with the unity e and $u\colon I_{\lambda_1} \to H_e$ a map. Then $I_h = \{s \in S \mid (s)h = 0_T\}$ is an ideal of S and the map $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ defined by the formulae*

$$((\alpha, s, \beta))\sigma = \begin{cases} ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi), & \text{if} \quad s \notin S \setminus I_h; \\ 0_2, & \text{if} \quad s \in I^*_h, \end{cases}$$

and $(0_1)\sigma = 0_2$ is a non-trivial homomorphism from $B^0_{\lambda_1}(S)$ into $B^0_{\lambda_2}(T)$. Moreover, if for the semigroup T the following assertions hold:

(i) *Every idempotent of T lies in the center of T; and*
(ii) *T has the $\mathcal{B}^*_{\lambda_1}$-property,*

then every non-trivial homomorphism from $B^0_{\lambda_1}(S)$ into $B^0_{\lambda_2}(T)$ can be constructed in this manner.

Proof. A simple verification shows that the set I_h is an ideal in S and that σ is a homomorphism from the semigroup $B^0_{\lambda_1}(S)$ into the semigroup $B^0_{\lambda_2}(T)$.

Let $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ be a non-trivial semigroup homomorphism. We fix $\alpha \in I_{\lambda_1}$. Since the homomorphism $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ is non-trivial, $((\alpha, 1_S, \alpha))\sigma$ is a non-zero idempotent of $B^0_{\lambda_2}(T)$, and hence $((\alpha, 1_S, \alpha))\sigma = (\alpha', e, \alpha')$ for some $e \in (E(T))^*$ and $\alpha' \in I_{\lambda_2}$. Let H_e be a maximal subgroup in T which contains e as a unity and let G_1 be the group of units of S. Therefore we have that $((G_1)_{\alpha,\alpha})\sigma \subseteq (H_e)_{\alpha',\alpha'}$.

Since $(\beta, 1_S, \alpha)(\alpha, 1_S, \alpha) = (\beta, 1_S, \alpha)$ for any $\beta \in I_{\lambda_1}$, we have

$$((\beta, 1_S, \alpha))\sigma = ((\beta, 1_S, \alpha))\sigma \cdot (\alpha', e, \alpha'),$$

and hence

$$((\beta,1_S,\alpha))\sigma=((\beta)\varphi,(\beta)u,\alpha'),$$

for some $(\beta)\varphi\in I_{\lambda_2}$ and $(\beta)u\in T$. Similarly, we have

$$((\alpha,1_S,\beta))\sigma=(\alpha',(\beta)v,(\beta)\psi),$$

for some $(\beta)\psi\in I_{\lambda_2}$ and $(\beta)v\in T$. Since $(\alpha,1_S,\beta)(\beta,1_S,\alpha)=(\alpha,1_S,\alpha)$, we have

$$(\alpha',e,\alpha')=((\alpha,1_S,\alpha))\sigma=(\alpha',(\beta)v,(\beta)\psi)\cdot((\beta)\varphi,(\beta)u,\alpha')=(\alpha',(\beta)v\cdot(\beta)u,\alpha'),$$

and hence $(\beta)\varphi=(\beta)\psi=\beta'\in I_{\lambda_2}$ and $(\beta)v\cdot(\beta)u=e$. Similarly, since $(\beta,1_S,\alpha)\cdot(\alpha,1_S,\beta)=(\beta,1_S,\beta)$, we see that the element

$$((\beta,1_S,\beta))\sigma=((\beta,1_S,\alpha)(\alpha,1_S,\beta))\sigma=(\beta',(\beta)v\cdot(\beta)u,\beta')$$

is an idempotent, and hence the element $f=(\beta)v\cdot(\beta)u$ is an idempotent of the semigroup T. Since idempotents of T lie in the center of T, we conclude that

$$\begin{aligned}(\alpha',e,\alpha')&=((\alpha,1_S,\alpha))\sigma=((\alpha,1_S,\beta)\cdot(\beta,1_S,\beta)\cdot(\beta,1_S,\alpha))\sigma=\\&=(\alpha',(\beta)v,\beta')\cdot(\beta',f,\beta')\cdot(\beta',(\beta)u,\alpha')=\\&=(\alpha',(\beta)v\cdot f\cdot(\beta)u,\alpha')=(\alpha',f\cdot(\beta)v\cdot(\beta)u,\alpha')=\\&=(\alpha',f\cdot e,\alpha')\end{aligned}$$

and

$$\begin{aligned}(\beta',f,\beta')&=((\beta,1_S,\beta))\sigma=((\beta,1_S,\alpha)\cdot(\alpha,1_S,\alpha)\cdot(\alpha,1_S,\beta))\sigma=\\&=(\beta',(\beta)u,\alpha')\cdot(\alpha',e,\alpha')\cdot(\alpha',(\beta)v,\beta')=\\&=(\beta',(\beta)u\cdot e\cdot(\beta)v,\alpha')=(\beta',(\beta)u\cdot(\beta)v\cdot e,\beta')=\\&=(\beta',f\cdot e,\beta'),\end{aligned}$$

and hence $e=f\cdot e=f$. Therefore $(\beta)v\cdot(\beta)u=(\beta)u\cdot(\beta)v=e$, $(\beta)v,(\beta)u\in H_e$, and hence $(\beta)v$ and $(\beta)u$ are inverse elements in the subgroup H_e. If $(\gamma)\varphi=(\delta)\varphi$ for $\gamma,\delta\in I_{\lambda_1}$ then

$$0_1\neq(\alpha',e,(\gamma)\varphi)\cdot((\delta)\varphi,e,\alpha')=((\alpha,1_S,\gamma))\sigma\cdot((\delta,1_S,\alpha))\sigma,$$

and since σ is a non-trivial homomorphism, we have $(\alpha,1_S,\gamma)\cdot(\delta,1_S,\alpha)\neq 0$ and hence $\gamma=\delta$. Thus $\varphi\colon I_{\lambda_1}\to I_{\lambda_2}$ is a one-to-one map.

Therefore for $s\in S\setminus I_h$ we have

$$\begin{aligned}((\gamma,s,\delta))\sigma&=((\gamma,1_S,\alpha)\cdot(\alpha,s,\alpha)\cdot(\alpha,1_S,\delta))\sigma=\\&=((\gamma,1_S,\alpha))\sigma\cdot((\alpha,s,\alpha))\sigma\cdot((\alpha,1_S,\delta))\sigma=\\&=((\gamma)\varphi,(\gamma)u,\alpha')\cdot(\alpha',(s)h,\alpha')\cdot(\alpha',((\delta)u)^{-1},(\delta)\varphi)=\\&=((\gamma)\varphi,(\gamma)u\cdot(s)h\cdot((\delta)u)^{-1},(\delta)\varphi).\end{aligned}$$

Corollary 3.9 implies that $((\alpha,s,\beta))\sigma=0_2$ for all $s\in I_h$ and by Proposition 3.4 we conclude that $(0_1)\sigma=0_2$. □

Remark 3.11. We observe that if a semigroup T satisfies one of the following conditions:

(i) T^* is a cancellative monoid; or

(ii) T is an inverse Clifford monoid,

then the second assertion of Theorem 3.10 holds. Also, Examples 3.12 and 3.13 imply that this statement is not true for inverse monoids with zeros and completely regular monoids with zeros.

Example 3.12. Let T be the bicyclic semigroup $\mathscr{C}(p,q) = \langle p,q \mid pq = 1\rangle$ with adjoined zero. Then we can write every element of the semigroup $\mathscr{C}(p,q)$ as q^ip^j for some $i,j = 0,1,2,\ldots$. We define a homomorphism $\sigma\colon B_2 \to B_2^0(\mathscr{C}(p,q))$ as follows:

$$\begin{aligned} &(0_1)\sigma = 0_2, && \\ &((1,1,1))\sigma = (1,1,1), && ((1,1,2))\sigma = (1,p,2), \\ &((2,1,2))\sigma = (2,qp,2), && ((2,1,1))\sigma = (2,q,1). \end{aligned}$$

Example 3.13. Let T be a 2×2-rectangular band with adjoined unity 1_T and zero 0_T, i. e. $T = \{(1,1),(1,2,),(2,1),(2,2),0_T\}$. We define a homomorphism $\sigma\colon B_2^0 \to B_2(T)$ as follows:

$$\begin{aligned} &(0_1)\sigma = 0_2, && \\ &((1,1,1))\sigma = (1,(1,1),1), && ((1,1,2))\sigma = (1,(1,2),2), \\ &((2,1,2))\sigma = (2,(2,2),2), && ((2,1,1))\sigma = (2,(2,1),1). \end{aligned}$$

We observe that a composition of two non-trivial homomorphisms of the Brandt λ-extensions of monoids with zeros may be the trivial homomorphism. This observation follows from the next example.

Example 3.14. Consider the set $E = \{a,b,c\}$ with the following semigroup operation:

$$a\cdot a = a, \qquad a\cdot b = b\cdot a = b \qquad \text{and} \qquad a\cdot c = c\cdot a = b\cdot c = c\cdot b = c\cdot c = c.$$

Then E with this operation is a semilattice with zero c and unity a, and hence the conditions $(i)-(ii)$ of Theorem 3.10 hold for the monoid E. We define a homomorphism $h\colon E\to E$ as follows: $(a)h = b$ and $(b)h = (c)h = c$. Then for any non-empty set I_λ the homomorphism $\sigma\colon B_\lambda^0(E) \to B_\lambda^0(E)$ defined by formulae

$$((\alpha,a,\beta))\sigma = (\alpha,b,\beta), \qquad ((\alpha,b,\beta))\sigma = 0, \qquad \text{and} \qquad (0)\sigma = 0,$$

where $\alpha,\beta\in I_\lambda$ and 0 is the zero of the semigroup $B_\lambda^0(E)$, the composition $\sigma\circ\sigma\colon B_\lambda^0(E)\to B_\lambda^0(E)$ is the trivial homomorphism.

Proposition 2.1(i) yields simple sufficient conditions that a composition of non-trivial homomorphisms of the Brandt λ^0-extensions is a non-trivial homomorphism:

Proposition 3.15. *Let λ_1, λ_2 and λ_3 be cardinals such that $\lambda_1\leqslant\lambda_2\leqslant\lambda_3$ and let S, T and R be monoids with zeros. Let $\sigma_1\colon B_{\lambda_1}^0(S)\to B_{\lambda_2}^0(T)$ and $\sigma_2\colon B_{\lambda_2}^0(T)\to B_{\lambda_3}^0(R)$ be non-trivial homomorphisms. If one of the following conditions holds*

- (i) *For some $\alpha\in I_{\lambda_1}$ the restriction $\sigma_1|_{S_{\alpha,\alpha}}\colon S_{\alpha,\alpha}\to(S_{\alpha,\alpha})\sigma_1\subset B_{\lambda_2}^0(T)$ of σ_1 is a monoid homomorphism; or*
- (ii) *$E(T) = \{1_T, 0_T\}$,*

then the composition $\sigma_1\circ\sigma_2\colon B_{\lambda_1}^0(S)\to B_{\lambda_3}^0(R)$ is a non-trivial homomorphism.

Since the semigroup of matrix units is congruence-free (cf. Theorem 1 of [8]), we get the following proposition:

Proposition 3.16. *Let λ_1, λ_2 and λ_3 be cardinals such that $\lambda_1\leqslant\lambda_2\leqslant\lambda_3$ and let S, T and R be monoids with zeros. Let $\sigma_1\colon B_{\lambda_1}^0(S)\to B_{\lambda_2}^0(T)$ and $\sigma_2\colon B_{\lambda_2}^0(T)\to B_{\lambda_3}^0(R)$ be non-trivial homomorphisms. Then the composition $\sigma_1\circ\sigma_2\colon B_{\lambda_1}^0(S)\to B_{\lambda_3}^0(R)$ is a non-trivial homomorphism if and only if $((\alpha,1_S,\beta))\sigma_1\notin\{(\alpha',t,\beta')\in B_{\lambda_2}^0(T)\mid((\alpha',t,\beta'))\sigma_2 = 0_3\}$, for some $\alpha,\beta\in I_{\lambda_1}$, $\alpha',\beta'\in I_{\lambda_2}$.*

4. The category of the Brandt λ^0-extensions of monoids with zeros

Let S and T be monoids with zeros. Let $\mathtt{Hom}_0(S,T)$ be the set of all homomorphisms $\sigma\colon S\to T$ such that $(0_S)\sigma=0_T$. We put

$$\mathbf{E}_1(S,T)=\{e\in E(T)\mid \text{there exists } \sigma\in\mathtt{Hom}_0(S,T) \text{ such that } (1_S)\sigma=e\}$$

and define the family

$$\mathscr{H}_1(S,T)=\{H(e)\mid e\in\mathbf{E}_1(S,T)\},$$

where we denote the maximal subgroup with the unity e in the semigroup T by $H(e)$. Also by $\mathfrak{B}$ we denote the class of monoids S with zeros such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

We define a category $\mathscr{B}$ as follows:

(*i*) $\mathbf{Ob}(\mathscr{B})=\{(S,I)\mid S\in\mathfrak{B}$ and I is a non-empty set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;

(*ii*) $\mathbf{Mor}(\mathscr{B})$ consists of triples $(h,u,\varphi)\colon(S,I)\to(S',I')$, where

$$\begin{aligned} &h\colon S\to S' \text{ is a homomorphism such that } h\in\mathtt{Hom}_0(S,S'),\\ &u\colon I\to H(e) \text{ is a map , for } H(e)\in\mathscr{H}_1(S,S'),\\ &\varphi\colon I\to I' \text{ is an one-to-one map,}\end{aligned}\tag{1}$$

with the composition

$$(h,u,\varphi)(h',u',\varphi')=(hh',[u,\varphi,h',u'],\varphi\varphi'),\tag{2}$$

where the map $[u,\varphi,h',u']\colon I\to H(e)$ is defined by the formula

$$(\alpha)[u,\varphi,h',u']=((\alpha)\varphi)u'\cdot((\alpha)u)h' \qquad \text{for } \alpha\in I.\tag{3}$$

A straightforward verification shows that $\mathscr{B}$ is a category with the identity morphism $\varepsilon_{(S,I)}=(\mathrm{Id}_S,u_0,\mathrm{Id}_I)$ for any $(S,I)\in\mathbf{Ob}(\mathscr{B})$, where $\mathrm{Id}_S\colon S\to S$ and $\mathrm{Id}_I\colon I\to I$ are identity maps and $(\alpha)u_0=1_S$ for all $\alpha\in I$.

We define the category $\mathscr{B}^*(\mathscr{S})$ as follows:

(*i*) $\mathbf{Ob}(\mathscr{B}^*(\mathscr{S}))$ are all Brandt λ^0-extensions of monoids S with zeros such that S has the $\mathcal{B}^*$-property and every idempotent of S lies in the center of S;

(*ii*) $\mathbf{Mor}(\mathscr{B}^*(\mathscr{S}))$ are homomorphisms of the Brandt λ^0-extensions of monoids S with zeros such that S has the $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

For each $(S,I_{\lambda_1})\in\mathbf{Ob}(\mathscr{B})$ with non-trivial S, let $\mathbf{B}(S,I_{\lambda_1})=B^0_{\lambda_1}(S)$ be the Brandt λ^0-extension of the semigroup S. For each $(h,u,\varphi)\in\mathbf{Mor}(\mathscr{B})$ with a non-trivial homomorphism h, where $(h,u,\varphi)\colon(S,I_{\lambda_1})\to(T,I_{\lambda_2})$ and $(T,I_{\lambda_2})\in\mathbf{Ob}(\mathscr{B})$, we define a map $\mathbf{B}(h,u,\varphi)\colon\mathbf{B}(S,I_{\lambda_1})=B^0_{\lambda_1}(S)\to\mathbf{B}(T,I_{\lambda_2})=B^0_{\lambda_2}(T)$ as follows:

$$((\alpha,s,\beta))[\mathbf{B}(h,u,\varphi)]=\begin{cases}((\alpha)\varphi,(\alpha)u\cdot(s)h\cdot((\beta)u)^{-1},(\beta)\varphi), & \text{if}\quad s\notin S\setminus I_h;\\ 0_2, & \text{if}\quad s\in I_h^*,\end{cases}\tag{4}$$

and $(0_1)[\mathbf{B}(h,u,\varphi)]=0_2$, where $I_h=\{s\in S\mid (s)h=0_T\}$ is an ideal of S and 0_1 and 0_2 are the zeros of the semigroups $B^0_{\lambda_1}(S)$ and $B^0_{\lambda_2}(T)$, respectively. For each $(h,u,\varphi)\in\mathbf{Mor}(\mathscr{B})$ with a trivial homomorphism h we define a map $\mathbf{B}(h,u,\varphi)\colon\mathbf{B}(S,I_{\lambda_1})=B^0_{\lambda_1}(S)\to\mathbf{B}(T,I_{\lambda_2})=B^0_{\lambda_2}(T)$ as follows: $(a)[\mathbf{B}(h,u,\varphi)]=$

0_2 for all $a \in \mathbf{B}(S, I_{\lambda_1}) = B^0_{\lambda_1}(S)$. If S is a trivial semigroup then we define $\mathbf{B}(S, I_{\lambda_1})$ to be a trivial semigroup.

A functor $\mathbf{F}$ from a category $\mathscr{C}$ into a category $\mathscr{K}$ is called *full* if for any $a, b \in \mathbf{Ob}(\mathscr{C})$ and for any $\mathscr{K}$-morphism $\alpha\colon \mathbf{F}a \to \mathbf{F}b$ there exists a $\mathscr{C}$-morphism $\beta\colon a \to b$ such that $\mathbf{F}\beta = \alpha$, and $\mathbf{F}$ called *representative* if for any $a \in \mathbf{Ob}(\mathscr{K})$ there exists $b \in \mathbf{Ob}(\mathscr{C})$ such that a and $\mathbf{F}b$ are isomorphic.

Theorem 4.1. *$\mathbf{B}$ is a full representative functor from $\mathscr{B}$ into $\mathscr{B}^*(\mathscr{S})$.*

Proof. For any $(S, I_\lambda) \in \mathbf{Ob}(\mathscr{B})$, $\mathbf{B}((S, I_\lambda))$ is the the Brandt λ^0-extension of the monoid with zero S by Proposition 3.2, and hence we have that $\mathbf{B}(S, I_\lambda) \in \mathbf{Ob}(\mathscr{B}^*(\mathscr{S}))$. By Theorem 3.10 we have that for a $\mathscr{B}$-morphism $(h, u, \varphi)\colon (S, I_{\lambda_1}) \to (T, I_{\lambda_2})$, $\mathbf{B}(h, u, \varphi)$ is a non-trivial homomorphism of $\mathbf{B}(S, I_{\lambda_1})$ into $\mathbf{B}(T, I_{\lambda_2})$ in the case when h is a non-trivial homomorphism. Obviously, $\mathbf{B}\varepsilon_{(S,I)} = \mathbf{B}(\mathrm{Id}_S, u_0, \mathrm{Id}_I)$ is the identity automorphism of $\mathbf{B}(S, I)$. Let $(h, u, \varphi)\colon (S, I_{\lambda_1}) \to (T, I_{\lambda_2})$ and $(f, v, \psi)\colon (T, I_{\lambda_2}) \to (R, I_{\lambda_3})$ be $\mathscr{B}$-morphisms with non-trivial h and f. Then for any $(\alpha, s, \beta) \in \mathbf{B}(S, I_{\lambda_1})$ we get

$$(\alpha, s, \beta)\,[\mathbf{B}(h, u, \varphi)]\big[\mathbf{B}(f, v, \psi)\big] =$$
$$= \begin{cases} \big(((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi)\big)\big[\mathbf{B}(f, v, \psi)\big], & \text{if} \quad s \notin S \setminus I_h; \\ (0_2)\big[\mathbf{B}(f, v, \psi)\big], & \text{if} \quad s \in I^*_h, \end{cases}$$

$$\big(((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi)\big)\big[\mathbf{B}(f, v, \psi)\big] =$$
$$= \begin{cases} \big(((\alpha)\varphi)\psi, ((\alpha)\varphi)v \cdot \big((\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}\big)f \cdot \big(((\beta)\psi)v\big)^{-1}, ((\beta)\varphi)\psi\big), & \text{if } (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1} \notin T \setminus I_f; \\ (0_2)\big[\mathbf{B}(f, v, \psi)\big], & \text{if } (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1} \in I^*_f, \end{cases}$$
$$= \begin{cases} \big(((\alpha)\varphi)\psi, \big(((\alpha)\varphi)v \cdot ((\alpha)u)f\big) \cdot ((s)h)f \cdot \big(((\beta)\varphi)v \cdot ((\beta)u)f\big)^{-1}, ((\beta)\varphi)\psi\big), & \text{if } (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1} \notin T \setminus I_f; \\ (0_2)\big[\mathbf{B}(f, v, \psi)\big], & \text{if } (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1} \in I^*_f, \end{cases}$$

and $(0_1)[\mathbf{B}(h, u, \varphi)]\big[\mathbf{B}(f, v, \psi)\big] = (0_2)\big[\mathbf{B}(f, v, \psi)\big] = 0_3$. In the case when for at least one of the $\mathscr{B}$-morphisms $(h, u, \varphi)\colon (S, I_{\lambda_1}) \to (T, I_{\lambda_2})$ and $(f, v, \psi)\colon (T, I_{\lambda_2}) \to (R, I_{\lambda_3})$, either h or f is trivial, we have by Proposition 3.1 that $(x)[\mathbf{B}(h, u, \varphi)]\big[\mathbf{B}(f, v, \psi)\big] = 0_3$. Therefore $\mathbf{B}$ preserves the compositions of morphisms, and hence $\mathbf{B}$ is a functor from $\mathscr{B}$ into $\mathscr{B}^*(\mathscr{S})$.

Theorem 3.10 implies that the functor $\mathbf{B}$ is full and by the definition of the Brandt λ^0-extension we conclude that the functor $\mathbf{B}$ is representative. □

We define the first series of categories $\mathscr{B}\mathscr{I}$, $\mathscr{B}\mathscr{I}\mathscr{C}$, $\mathscr{B}\mathscr{S}\mathscr{L}$, $\mathscr{B}\mathscr{S}_2$ and $\mathscr{B}\mathscr{G}$ as follows:

(*i*) $\mathbf{Ob}(\mathscr{B}\mathscr{I}) = \{(S, I) \mid S \in \mathfrak{B}$ is an inverse monoid and I is a non-empty set$\}$, and if S is a trivial semigroup then we identify (S, I) and (S, J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{B}\mathscr{I}\mathscr{C}) = \{(S, I) \mid S$ is an inverse Clifford monoid and I is a non-empty set$\}$, and if S is a trivial semigroup then we identify (S, I) and (S, J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{B}\mathscr{S}\mathscr{L}) = \{(S, I) \mid S$ is a semilattice with unity and zero and I is a non-empty set$\}$, and if S is a trivial semigroup then we identify (S, I) and (S, J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{B}\mathscr{S}_2) = \{(S, I) \mid S$ is a monoid with two idempotents, the zero and the unity and I is a non-empty set$\}$;
$\mathbf{Ob}(\mathscr{B}\mathscr{G}) = \{(S, I) \mid S$ is a group and I is a non-empty set$\}$;

(*ii*) $\mathbf{Mor}(\mathscr{BI})$, $\mathbf{Mor}(\mathscr{BIC})$, $\mathbf{Mor}(\mathscr{BSL})$, $\mathbf{Mor}(\mathscr{BS}_2)$ and $\mathbf{Mor}(\mathscr{BG})$ consist of corresponding triples $(h, u, \varphi)\colon (S, I) \to (S', I')$, which satisfy condition (1).

Obviously, $\mathscr{BI}$, $\mathscr{BIC}$, $\mathscr{BSL}$, $\mathscr{BS}_2$ and $\mathscr{BG}$ are subcategories of the category $\mathscr{B}$.

The second series of categories $\mathscr{B}^*(\mathscr{IS})$, $\mathscr{B}^*(\mathscr{ICS})$, $\mathscr{B}^*(\mathscr{SL})$, $\mathscr{B}^*(\mathscr{S}_2)$ and $\mathscr{B}^*(\mathscr{G})$ is defined as follows:

(*i*) $\mathbf{Ob}(\mathscr{B}^*(\mathscr{IS}))$ are all Brandt λ^0-extensions of inverse monoids S with zeros such that S has $\mathfrak{B}^*$-property and every idempotent of S lies in the center of S;
$\mathbf{Ob}(\mathscr{B}^*(\mathscr{ICS}))$ are all Brandt λ^0-extensions of inverse Clifford monoids with zeros;
$\mathbf{Ob}(\mathscr{B}^*(\mathscr{SL}))$ are all Brandt λ^0-extensions of semilattices with zeros and identities;
$\mathbf{Ob}(\mathscr{B}^*(\mathscr{S}_2))$ are all Brandt λ^0-extensions of monoids with two idempotents zeros and identities;
$\mathbf{Ob}(\mathscr{B}^*(\mathscr{G}))$ are all Brandt semigroups;

(*ii*) $\mathbf{Mor}(\mathscr{B}^*(\mathscr{IS}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{ICS}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{SL}))$ and $\mathbf{Mor}(\mathscr{B}^*(\mathscr{S}_2))$ are homomorphisms of the Brandt λ^0-extensions of monoids S with zeros such that S has the $\mathcal{B}^*$-property and every idempotent of S lies in the center of S, inverse Clifford monoids with zeros, semilattices with zeros and identities, monoids with two idempotents zeros and identities and $\mathbf{Mor}(\mathscr{B}^*(\mathscr{G}))$ be non-trivial homomorphisms of Brandt semigroups.

The proof of the following proposition is similar to the proof of Theorem 4.1.

Proposition 4.2. *$\boldsymbol{B}$ is a full representative functor from $\mathscr{BI}$ [resp., $\mathscr{BIC}$, $\mathscr{BSL}$, $\mathscr{BS}_2$ and $\mathscr{BG}$] into $\mathscr{B}^*(\mathscr{IS})$ [resp., $\mathscr{B}^*(\mathscr{ICS})$, $\mathscr{B}^*(\mathscr{SL})$, $\mathscr{B}^*(\mathscr{S}_2)$ and $\mathscr{B}^*(\mathscr{G})$].*

Proposition 4.3. *Let $(h, u, \varphi)\colon (S, I_{\lambda_1}) \to (T, I_{\lambda_2})$ and $(f, v, \psi)\colon (S, I_{\lambda_1}) \to (T, I_{\lambda_2})$ be $\mathscr{BSL}$-morphisms. Then $\boldsymbol{B}(h, u, \varphi) = \boldsymbol{B}(f, v, \psi)$ if and only if $h = f$, $u = v$ and $\varphi = \psi$.*

Proof. By definition of the functor $\mathbf{B}$ we have $\mathbf{B}(h, u, \varphi) = \mathbf{B}(f, v, \psi)$ if and only if

$$((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi) = ((\alpha)\psi, (\alpha)v \cdot (s)f \cdot ((\beta)v)^{-1}, (\beta)\psi),$$

for $(\alpha, s, \beta) \in \mathbf{B}(S, I_{\lambda_1})$. Then $\varphi = \psi$ and since for semilattices S we have $(\alpha)u = ((\alpha)u)^{-1} = (1_S)h$ and $(\alpha)v = ((\alpha)v)^{-1} = (1_S)f$, we get that $h = f$. $\square$

Remark 4.4. The definition of the functor $\mathbf{B}$ implies that $\mathbf{B}$ is one-to-one on objects of the category $\mathscr{BI}$ [resp., $\mathscr{BIC}$, $\mathscr{BSL}$, $\mathscr{BS}_2$ and $\mathscr{BG}$]. Also, Proposition 4.3 implies that the functor $\mathbf{B}$ is one-to-one on morphisms of the category $\mathscr{BSL}$, but Proposition II.3.9 of [15] implies that the functor $\mathbf{B}$ is not one-to-one on morphisms of the category $\mathscr{BI}$ [resp., $\mathscr{BIC}$, $\mathscr{BS}_2$ and $\mathscr{BG}$].

Therefore Theorem 4.1 and Proposition 4.3 imply:

Corollary 4.5. *The categories $\mathscr{BSL}$ and $\mathscr{B}^*(\mathscr{SL})$ are isomorphic.*

5. Topological Brandt λ^0-extensions of topological monoids with zero

A topological space S which is algebraically a semigroup with a jointly continuous semigroup operation is called a *topological semigroup*. A *topological inverse semigroup* is a topological semigroup S that is algebraically an inverse semigroup with continuous inversion. If τ is a topology on a (inverse) semigroup S such that (S,τ) is a topological (inverse) semigroup, then τ is called a (*inverse*) *semigroup topology* on S.

In this section we shall follow the terminology of [2] and [7].

Definition 5.1 ([12])**.** Let $\mathscr{S}$ be some class of topological monoids with zero. Let λ be any cardinal $\geqslant 1$, and $(S,\tau)\in\mathscr{S}$. Let τ_B be a topology on $B^0_\lambda(S)$ such that:

a) $\left(B^0_\lambda(S),\tau_B\right)\in\mathscr{S}$; and

b) $\tau_B|_{S_{\alpha,\alpha}}=\tau$ for some $\alpha\in I_\lambda$.

Then $\left(B^0_\lambda(S),\tau_B\right)$ is called the *topological Brandt λ^0-extension of* (S,τ) *in* $\mathscr{S}$. If $\mathscr{S}$ coincides with the class of all topological semigroups, then $\left(B^0_\lambda(S),\tau_B\right)$ is called the *topological Brandt λ^0-extension of* (S,τ).

Results of Section 2 of [12] imply that for any infinite cardinal λ and every non-trivial topological semigroup S, there are many topological Brandt λ^0-extensions of S, and for any topological inverse semigroup T, there are many topological Brandt λ^0-extensions of T in the class of topological inverse semigroups. Moreover, for any infinite cardinal λ on the Brandt λ^0-extension of two-element monoid with zero (i. e., on the infinite semigroup of $I_\lambda\times I_\lambda$-units) there exist many semigroup and inverse semigroup topologies (cf. [11]).

These observations imply that for infinite cardinals λ there does not exist a proposition for topological semigroups similar to Theorem 4.1 and Propositions 4.2 and 4.3. In this section we prove such statements for any finite non-zero cardinals.

Proposition 5.2. *Let λ be any finite non-zero cardinal. Let be (S,τ) a topological semigroup and τ_B a topology on $B^0_\lambda(S)$ such that $\left(B^0_\lambda(S),\tau_B\right)$ is a topological semigroup and $\tau_B|_{S_{\alpha,\alpha}}=\tau$ for some $\alpha\in I_\lambda$. Then the following assertions hold:*

(*i*) *If a non-empty subset $A\not\ni 0_S$ of S is open in S, then so is $A_{\beta,\gamma}$ in $\left(B^0_\lambda(S),\tau_B\right)$ for any $\beta,\gamma\in I_\lambda$;*

(*ii*) *If a non-empty subset $A\ni 0_S$ of S is open in S, then so is $\bigcup_{\beta,\gamma\in I_\lambda}A_{\beta,\gamma}$ in $\left(B^0_\lambda(S),\tau_B\right)$;*

(*iii*) *If a non-empty subset $A\not\ni 0_S$ of S is closed in S, then so is $A_{\beta,\gamma}$ in $\left(B^0_\lambda(S),\tau_B\right)$ for any $\beta,\gamma\in I_\lambda$;*

(*iv*) *If a non-empty subset $A\ni 0_S$ of S is closed in S, then so is $\bigcup_{\beta,\gamma\in I_\lambda}A_{\beta,\gamma}$ in $\left(B^0_\lambda(S),\tau_B\right)$;*

(*v*) *If x is a non-zero element of S and $\mathscr{B}_x$ is a base of the topology τ at x, then the family $\mathscr{B}_{(\beta,x,\gamma)}=\{U_{\beta,\gamma}\mid U\in\mathscr{B}_x\}$ is a base of the topology τ_B at the point $(\beta,x,\gamma)\in B^0_\lambda(S)$ for any $\beta,\gamma\in I_\lambda$;*

(*vi*) *If $\mathscr{B}_{0_S}$ is a base of the topology τ at zero 0_S of S, then the family $\mathscr{B}_0=\left\{\bigcup_{\beta,\gamma\in I_\lambda}U_{\beta,\gamma}\mid U\in\mathscr{B}_{0_S}\right\}$ is a base of the topology τ_B at zero 0 of the semigroup $B^0_\lambda(S)$.*

Proof. (*i*) Let $W\not\ni 0$ be an open set in $\left(B^0_\lambda(S),\tau_B\right)$ such that $W\cap S_{\alpha,\alpha}\in\tau_B|_{S_{\alpha,\alpha}}$. Suppose that $W\nsubseteq S^*_{\alpha,\alpha}$. We fix $(\alpha,x,\alpha)\in W$. Since $(\alpha,1_S,\alpha)\cdot(\alpha,x,\alpha)\cdot(\alpha,1_s,\alpha)=$

(α, x, α), there exists an open neighbourhood U of the point (α, x, α) such that $U \subseteq W$ and $(\alpha, 1_S, \alpha) \cdot U \cdot (\alpha, 1_s, \alpha) \subseteq W$. If $U \nsubseteq S^*_{\alpha,\alpha}$, then $0 \in (\alpha, 1_S, \alpha) \cdot U \cdot (\alpha, 1_s, \alpha) \subseteq W$, a contradiction. Therefore for any $(\alpha, x, \alpha) \in W$ there exists an open neighbourhood of (α, x, α) such that $U \subseteq S^*_{\alpha,\alpha}$, and hence $W \cap S^*_{\alpha,\alpha}$ is an open subset in $\big(B^0_\lambda(S), \tau_B\big)$.

By Definition 5.1 the set $A_{\alpha,\alpha}$ is open for some $\alpha \in I_\lambda$. Since the map $\varphi^{\alpha\alpha}_{\gamma\delta} \colon B^0_\lambda(S) \to B^0_\lambda(S)$ defined by the formula $(x)\varphi^{\alpha\alpha}_{\gamma\delta} = (\alpha, 1_S, \gamma) \cdot x \cdot (\delta, 1_S, \alpha)$ is continuous, we get that the set $A_{\gamma,\delta} = (A_{\alpha,\alpha})\big(\varphi^{\alpha\alpha}_{\gamma\delta}\big)^{-1}$ is open in $\big(B^0_\lambda(S), \tau_B\big)$ for any $\gamma, \delta \in I_\lambda$.

Let $A \ni 0$ be an open subset in S and W an open subset in $\big(B^0_\lambda(S), \tau_B\big)$ such that $W \cap S_{\alpha,\alpha} = A_{\alpha,\alpha}$ for some $\alpha \in I_\lambda$. Since the map $\varphi^{\alpha\alpha}_{\gamma\delta}$ is continuous for any $\alpha, \gamma, \delta \in I_\lambda$, the set

$$\widetilde{A}_{\gamma,\delta} = \bigcup_{(\iota,\kappa)\in(I_\lambda\times I_\lambda)\setminus(\gamma,\delta)} S_{\iota,\kappa} \cup A_{\gamma,\delta}$$

is an open subset in $\big(B^0_\lambda(S), \tau_B\big)$. Then since the set I_λ is finite, we have that $\bigcup\limits_{\alpha,\beta\in I_\lambda} A_{\alpha,\beta} = \bigcap\limits_{\gamma,\delta\in I_\lambda} \widetilde{A}_{\gamma,\delta}$ and this implies statement (ii).

Statements $(iii) - (vi)$ follow from (i) and (ii). □

Remark 5.3. Note that the statements (i), (iii), (iv) and (v) of Proposition 5.2 hold for any infinite cardinal λ. However, Example 21 and Proposition 25 of [11] imply that the statements (ii) and (vi) are false for any infinite cardinal λ.

We shall need the following lemma from [12]:

Lemma 5.4 ([12, Lemma 1]). *Let $\lambda \geqslant 2$ be any cardinal and $B^0_\lambda(S)$ the topological Brandt λ^0-extension of a topological monoid S with zero. Let T be a topological semigroup and $h \colon B^0_\lambda(S) \to T$ be a continuous homomorphism. Then the sets $(A_{\alpha\beta})h$ and $(A_{\gamma\delta})h$ are homeomorphic in T for all $\alpha, \beta, \gamma, \delta \in I_\lambda$, and all $A \subseteq S$.*

Proof. If h is an annihilating homomorphism, then the statement of the lemma is trivial.

Otherwise, we fix arbitrary $\alpha, \beta, \gamma, \delta \in I_\lambda$ and define the maps $\varphi^{\gamma\delta}_{\alpha\beta} \colon T \to T$ and $\varphi^{\alpha\beta}_{\gamma\delta} \colon T \to T$ by the formulae

$$(s)\varphi^{\gamma\delta}_{\alpha\beta} = ((\gamma, 1, \alpha))h \cdot s \cdot ((\beta, 1, \delta))h \qquad \text{and} \qquad (s)\varphi^{\alpha\beta}_{\gamma\delta} = ((\alpha, 1, \gamma))h \cdot s \cdot ((\delta, 1, \beta))h,$$

$s \in T$. Obviously,

$$\left(\left(((\alpha, x, \beta))\, h\right)\varphi^{\gamma\delta}_{\alpha\beta}\right) \varphi^{\alpha\beta}_{\gamma\delta} = ((\alpha, x, \beta))\, h \qquad \text{and} \qquad \left(\left(((\gamma, x, \delta))\, h\right)\varphi^{\alpha\beta}_{\gamma\delta}\right) \varphi^{\gamma\delta}_{\alpha\beta} = ((\gamma, x, \delta))\, h,$$

for all $\alpha, \beta, \gamma, \delta \in I_\lambda$, $x \in S^1$, and hence $\varphi^{\gamma\delta}_{\alpha\beta}\,|_{A_{\alpha\beta}} = (\varphi^{\alpha\beta}_{\gamma\delta})^{-1}\,|_{A_{\alpha\beta}}$. Since the maps $\varphi^{\gamma\delta}_{\alpha\beta}$ and $\varphi^{\alpha\beta}_{\gamma\delta}$ are continuous on T, the map $\varphi^{\gamma\delta}_{\alpha\beta}\,|_{h(A_{\alpha\beta})} \colon h(A_{\alpha\beta}) \to h(A_{\gamma\delta})$ is a homeomorphism. □

Proposition 5.5. *Let $\lambda \geqslant 1$ be any cardinal and $B^0_\lambda(S)$ the topological Brandt λ^0-extension of a topological (inverse) monoid S with zero in the class of topological (inverse) semigroups $\mathfrak{S}$. Let $T \in \mathfrak{S}$ and $h \colon B^0_\lambda(S) \to T$ be a continuous homomorphism. Then the image $(B^0_\lambda(S))h$ is the topological Brandt λ^0-extension of some monoid $M \in \mathfrak{S}$ with zero in the class $\mathfrak{S}$.*

Proof. Proposition 3.2 implies the algebraic part of the proposition. Since a subsemigroup of a topological semigroup is a topological semigroup, Lemma 5.4 implies that $(B^0_\lambda(S))h$ is the topological Brandt λ^0-extension of $(S_{\alpha,\alpha})h$ for some $a \in I_\lambda$. Also if S and T are topological inverse semigroups, then by Proposition II.2 of [6] the image $(B^0_\lambda(S))h$ is a topological inverse semigroup. □

Corollary 3.3 and Proposition 5.5 imply:

Corollary 5.6. *Let $B_\lambda(G)$ be a topological (inverse) Brandt semigroup. Let T be a topological (inverse) semigroup and $h\colon B_\lambda(G) \to T$ be a continuous homomorphism. Then the image $(B_\lambda(G))h$ is a topological (inverse) Brandt semigroup.*

Proposition 5.2 and Lemma 5.4 imply the following:

Lemma 5.7. *For any topological monoid (S,τ) with zero and for any finite cardinal $\lambda \geqslant 1$ there exists a unique topological Brandt λ^0-extension $\left(B^0_\lambda(S),\tau_B\right)$ and the topology τ_B generated by the base $\mathscr{B}_B = \bigcup\{\mathscr{B}_B(t) \mid t \in B^0_\lambda(S)\}$, where:*

- *(i) $\mathscr{B}_B(t) = \{(U(s))_{\alpha,\beta} \setminus \{0_S\} \mid U(s) \in \mathscr{B}_S(s)\}$, when $t = (\alpha,s,\beta)$ is a non-zero element of $B^0_\lambda(S)$, $\alpha,\beta \in I_\lambda$;*
- *(ii) $\mathscr{B}_B(0) = \{\bigcup_{\alpha,\beta\in I_\lambda}(U(0_S))_{\alpha,\beta} \mid U(0_S) \in \mathscr{B}_S(0_S)\}$, when 0 is the zero of $B^0_\lambda(S)$,*

and $\mathscr{B}_S(s)$ is a base of the topology τ at the point $s \in S$.

Moreover, if $\lambda \geqslant 1$ is any finite cardinal then a topological monoid (S,τ) with zero is a topological inverse semigroup if and only if $\left(B^0_\lambda(S),\tau_B\right)$ is the topological Brandt λ^0-extension of (S,τ) in the class of topological inverse semigroups.

The topological Brandt λ^0-extension $\left(B^0_\lambda(S),\tau_B\right)$ is called *compact* (resp., *countably compact*) if the topological space $\left(B^0_\lambda(S),\tau_B\right)$ is compact (resp., countably compact).

Propositions 5.8 and 5.9 describe the structures of compact Brandt λ^0-extensions and countably compact Brandt λ^0-extensions in the class of topological inverse semigroups.

Proposition 5.8. *A topological Brandt λ^0-extension $B^0_\lambda(S)$ of a topological monoid (S,τ) with zero is compact if and only if the cardinal $\lambda \geqslant 1$ is finite and (S,τ) is a compact topological semigroup. Moreover, for any compact topological monoid (S,τ) with zero and for any finite cardinal $\lambda \geqslant 1$ there exists a unique compact topological Brandt λ^0-extension $\left(B^0_\lambda(S),\tau_B\right)$ and the topology τ_B generated by the base $\mathscr{B}_B = \bigcup\{\mathscr{B}_B(t) \mid t \in B^0_\lambda(S)\}$, where:*

- *(i) $\mathscr{B}_B(t) = \{(U(s))_{\alpha,\beta} \setminus \{0_S\} \mid U(s) \in \mathscr{B}_S(s)\}$, when $t = (\alpha,s,\beta)$ is a non-zero element of $B^0_\lambda(S)$, $\alpha,\beta \in I_\lambda$;*
- *(ii) $\mathscr{B}_B(0) = \{\bigcup_{\alpha,\beta\in I_\lambda}(U(0_S))_{\alpha,\beta} \mid U(0_S) \in \mathscr{B}_S(0_S)\}$, when 0 is the zero of $B^0_\lambda(S)$,*

and $\mathscr{B}_S(s)$ is a base of the topology τ at the point $s \in S$.

Proof. Since by Theorem 10 of [11], the infinite semigroup of matrix units does not embed into a compact topological semigroup, the compactness of the topological Brandt λ^0-extension $\left(B^0_\lambda(S),\tau_B\right)$ of a topological semigroup (S,τ) implies that the cardinal λ is finite. Then by Theorem 1.7(e) of [2, Vol. 1], $(\alpha,1_S,\alpha)B^0_\lambda(S)(\alpha,1_S,\alpha) = S_{\alpha,\alpha}$ is a compact semigroup for any $\alpha \in I_\lambda$, and hence (S,τ) is a compact topological semigroup. The converse follows from Lemma 5.4 and the assertion that the finite union of compact spaces is a compact space.

Lemma 5.7 implies the last assertion of the proposition. □

Proposition 5.9. *The topological Brandt λ^0-extension $B^0_\lambda(S)$ of a topological monoid (S,τ) with zero in the class of topological inverse semigroups is countably compact if and only if the cardinal $\lambda \geqslant 1$ is finite and (S,τ) is a countably compact topological inverse semigroup. Moreover, for any countably compact topological monoid (S,τ) with zero and for any finite cardinal $\lambda \geqslant 1$ there exists a unique compact topological Brandt λ^0-extension $\left(B^0_\lambda(S),\tau_B\right)$ in the class of topological inverse semigroups and the topology τ_B generated by the base $\mathscr{B}_B = \bigcup\{\mathscr{B}_B(t) \mid t \in B^0_\lambda(S)\}$, where:*

- *(i)* *$\mathscr{B}_B(t) = \{(U(s))_{\alpha,\beta} \setminus \{0_S\} \mid U(s) \in \mathscr{B}_S(s)\}$, when $t = (\alpha, s, \beta)$ is a non-zero element of $B^0_\lambda(S)$, $\alpha, \beta \in I_\lambda$; and*
- *(ii)* *$\mathscr{B}_B(0) = \{\bigcup_{\alpha,\beta\in I_\lambda}(U(0_S))_{\alpha,\beta} \mid U(0_S) \in \mathscr{B}_S(0_S)\}$, when 0 is the zero of $B^0_\lambda(S)$,*

and $\mathscr{B}_S(s)$ is a base of the topology τ at the point $s \in S$.

Proof. By Theorem 14 of [11], the semigroup of $I_\lambda \times I_\lambda$-matrix units is a closed subsemigroup of any topological inverse semigroup T which contains it. By Theorem 6 of [11], on the infinite semigroup of matrix units there exists no countably compact inverse semigroup topology. Therefore λ is a finite cardinal, hence by Theorem 1.7(e) of [2, Vol. 1] and Theorem 3.10.4 of [7], $(\alpha, 1_S, \alpha)B^0_\lambda(S)(\alpha, 1_S, \alpha) = S_{\alpha,\alpha}$ is a countably compact topological semigroup for any $\alpha \in I_\lambda$, and thus (S,τ) is a countably compact topological semigroup. The converse follows from Lemma 5.4 and the assertion that the finite union of countably compact spaces is a countable compact space.

The last assertion of the proposition follows from Lemma 5.7. □

Theorem 5.10. *Let λ_1 and λ_2 be any finite cardinals such that $\lambda_2 \geqslant \lambda_1 \geqslant 1$. Let $B^0_{\lambda_1}(S)$ and $B^0_{\lambda_2}(T)$ be topological Brandt λ^0_1- and λ^0_2-extensions of topological monoids S and T with zero, respectively. Let $h\colon S \to T$ be a continuous homomorphism such that $(0_S)h = 0_T$ and $\varphi\colon I_{\lambda_1} \to I_{\lambda_2}$ an one-to-one map. Let e be a non-zero idempotent of T, H_e a maximal subgroup of T with unity e and $u\colon I_{\lambda_1} \to H_e$ a map. Then $I_h = \{s \in S \mid (s)h = 0_T\}$ is a closed ideal of S and the map $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ defined by the formulae*

$$((\alpha, s, \beta))\sigma = \begin{cases} ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi), & \text{if} \quad s \notin S \setminus I_h; \\ 0_2, & \text{if} \quad s \in I^*_h, \end{cases}$$

and $(0_1)\sigma = 0_2$, is a non-trivial continuous homomorphism from $B^0_{\lambda_1}(S)$ into $B^0_{\lambda_2}(T)$. Moreover if for the semigroup T the following conditions hold:

- *(i)* *Every idempotent of T lies in the center of T; and*
- *(ii)* *T has $\mathcal{B}^*_{\lambda_1}$-property,*

then every non-trivial continuous homomorphism from $B^0_{\lambda_1}(S)$ into $B^0_{\lambda_2}(T)$ can be so constructed.

Proof. The algebraic part of the theorem follows from Theorem 4.1.

Since the homomorphism h is continuous, $I_h = (0_T)h^{-1}$ is a closed ideal of the topological semigroup S.

Further we shall show that the homomorphism σ is continuous whenever is also h. We consider the following cases:

- *(i)* $(0_1)\sigma = 0_2$;

(*ii*) $((\alpha, s, \beta))\sigma = 0_2$, i. e. $s \in I_h$; and

(*iii*) $((\alpha, s, \beta))\sigma = ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi)$,

where (α, s, β) is any non-zero element of the semigroup $B^0_{\lambda_1}(S)$.

Without loss of generality we may assume that $\varphi\colon I_{\lambda_1} \to I_{\lambda_2}$ is a bijection. Moreover, for the simplification of the proof we can assume that $(\alpha)\varphi = \alpha$ for all $\alpha \in I_{\lambda_1}$.

Consider case (*i*). Let $U(0_2) = \bigcup_{\alpha,\beta \in I_{\lambda_2}} (U(0_T))_{\alpha,\beta}$ be any open basic neighbourhood of the zero 0_2 in $B^0_{\lambda_2}(T)$. Since left and right translations in T and the homomorphism $h\colon S \to T$ are continuous maps, there exists for any $(\alpha)u, ((\beta)u)^{-1} \in B^0_{\lambda_1}(S)$, an open neighbourhood $V^{\alpha,\beta}(0_S)$ in S such that $(\alpha)u \cdot (V^{\alpha,\beta}(0_S))h \cdot ((\beta)u)^{-1} \subseteq U(0_T)$. We put $V(0_S) = \bigcap_{\alpha,\beta \in I_{\lambda_1}} V^{\alpha,\beta}(0_S)$ and $V(0_1) = \bigcup_{\alpha,\beta \in I_{\lambda_1}} (V(0_S))_{\alpha,\beta}$. Then $(V(0_1))\sigma \subseteq U(0_2)$.

In case (*ii*) we have that $(s)h = 0_T$. Let $U(0_2) = \bigcup_{\alpha,\beta \in I_{\lambda_2}} (U(0_T))_{\alpha,\beta}$ be any basic open neighbourhood of the zero 0_2 in $B^0_{\lambda_2}(T)$. Since left and right translations in T and the homomorphism $h\colon S \to T$ are continuous maps, for the open neighbourhood $U(0_T)$ of the zero 0_T there exists an open neighbourhood $V(s)$ of s in S such that $(\alpha)u \cdot (V(s))h \cdot ((\beta)u)^{-1} \subseteq U(0_T)$. Therefore we have that $((V(s))_{\alpha,\beta})\sigma \subseteq U(0_2)$.

Next we consider case (*iii*). Let $U_{\alpha,\beta}$ be any basic open neighbourhood of the element $((\alpha, s, \beta))\sigma = (\alpha, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, \beta)$ in $B^0_{\lambda_2}(T)$. Since left and right translations in the semigroup T and the homomorphism $h\colon S \to T$ are continuous maps, there exists an open neighbourhood $V(s)$ of the point s in S such that $(\alpha)u \cdot (V(s))h \cdot ((\beta)u)^{-1} \subseteq U$ and hence we get $((V(s))_{\alpha,\beta})\sigma \subseteq U_{\alpha,\beta}$.

Since left and right translations in the topological semigroup $B^0_{\lambda_2}(T)$ are continuous and any restriction of a continuous map is a continuous map, the continuity of the homomorphism $\sigma\colon B^0_{\lambda_1}(S) \to B^0_{\lambda_2}(T)$ implies the continuity of h. □

Note that the statements of Theorem 5.10 are false for the topological Brandt λ^0-extensions when λ is an infinite cardinal. This follows from the next example:

Example 5.11. Let λ be an infinite cardinal. On B_λ we define a topology τ_{mi} as follows: all non-zero elements of B_λ are isolated points and the family

$$\mathscr{B}(0) = \{V_{\alpha_1} \cap \cdots \cap V_{\alpha_i} \cap H_{\beta_1} \cap \cdots \cap H_{\beta_j} \mid \alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_j \in I_\lambda, i, j \in \mathbb{N}\},$$

where $V_\gamma = B_\lambda \setminus \{(\gamma, \delta) \mid \delta \in I_\lambda\}$ and $H_\nu = B_\lambda \setminus \{(\delta, \nu) \mid \delta \in I_\lambda\}$, $\gamma, \nu \in I_\lambda$, determined a base of the topology τ_{mi} at zero of B_λ (cf. [11]). By Proposition 25 of [11], (B_λ, τ_{mi}) is a topological inverse semigroup.

Let $\mathfrak{d}$ be the discrete topology on B_λ. Then the identity map $\sigma\colon (B_\lambda, \tau_{mi}) \to (B_\lambda, \mathfrak{d})$ is not continuous, but the maps h, φ and u are as requested in the statements of Theorem 5.10.

Similarly to Section 4, we define new categories of topological semigroups and pairs of finite sets and topological semigroups.

Let S and T be topological monoids with zeros. Let be $\mathtt{CHom}_0(S, T)$ be the set of all continuous homomorphisms $\sigma\colon S \to T$ such that $(0_S)\sigma = 0_T$. We put

$$\mathbf{E}_1^{top}(S, T) = \{e \in E(T) \mid \text{there exists } \sigma \in \mathtt{CHom}_0(S, T) \text{ such that } (1_S)\sigma = e\}$$

and define the family

$$\mathscr{H}_1^{top}(S, T) = \{H(e) \mid e \in \mathbf{E}_1^{top}(S, T)\},$$

where by $H(e)$ we denote the maximal subgroup with the unity e in the semigroup T. Also, by $\mathfrak{TB}$ we denote the class of all topological monoids S with zero such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

We define the category $\mathscr{TB}_{\mathrm{fin}}$ as follows:

(i) $\mathbf{Ob}(\mathscr{TB}_{\mathrm{fin}}) = \{(S,I) \mid S \in \mathfrak{TB}$ and I is a finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all finite sets I and J;

(ii) $\mathbf{Mor}(\mathscr{TB}_{\mathrm{fin}})$ consists of triples $(h,u,\varphi)\colon (S,I) \to (S',I')$, where

$$\begin{aligned} &h\colon S \to S' \text{ is a continuous homomorphism such that } h \in \mathtt{CHom}_0(S,S'),\\ &u\colon I \to H(e) \text{ is a map , for } H(e) \in \mathscr{H}_1^{top}(S,S'),\\ &\varphi\colon I \to I' \text{ is an one-to-one map,}\end{aligned} \tag{5}$$

with the composition defined by formulae (2) and (3).

Straightforward verification shows that $\mathscr{TB}_{\mathrm{fin}}$ is the category with the identity morphism $\varepsilon_{(S,I)} = (\mathrm{Id}_S, u_0, \mathrm{Id}_I)$ for any $(S,I) \in \mathbf{Ob}(\mathscr{TB}_{\mathrm{fin}})$, where $\mathrm{Id}_S\colon S \to S$ and $\mathrm{Id}_I\colon I \to I$ are identity maps and $(\alpha)u_0 = 1_S$ for all $\alpha \in I$.

We define a category $\mathscr{B}^*_{\mathrm{fin}}(\mathscr{TS})$ as follows:

(i) Let $\mathbf{Ob}(\mathscr{B}^*_{\mathrm{fin}}(\mathscr{TS}))$ be all finite topological Brandt λ^0-extensions of topological monoids S with zeros such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S;

(ii) Let $\mathbf{Mor}(\mathscr{B}^*_{\mathrm{fin}}(\mathscr{TS}))$ be homomorphisms of finite topological Brandt λ^0-extensions of topological monoids S with zeros such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S.

We define a functor $\mathbf{B}$ from the category $\mathscr{TB}_{\mathrm{fin}}$ into the category $\mathscr{B}^*_{\mathrm{fin}}(\mathscr{TS})$ similarly as in Section 4 (cf. 4). Theorems 4.1 and 5.10 imply:

Theorem 5.12. *$\boldsymbol{B}$ is a full representative functor from $\mathscr{TB}_{\mathrm{fin}}$ into $\mathscr{B}^*_{\mathrm{fin}}(\mathscr{TS})$.*

We define the first series of categories as follows:

(i) $\mathbf{Ob}(\mathscr{TB}_{\mathrm{fin}}\mathscr{I}) = \{(S,I) \mid S \in \mathfrak{B}$ is a topological inverse monoid and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{I}) = \{(S,I) \mid S \in \mathfrak{B}$ is a countably compact topological inverse monoid and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{I}) = \{(S,I) \mid S \in \mathfrak{B}$ is a compact topological inverse monoid and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{TB}_{\mathrm{fin}}\mathscr{IC}) = \{(S,I) \mid S$ is a topological inverse Clifford monoid and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{IC}) = \{(S,I) \mid S$ is a countably compact topological inverse Clifford monoid and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{IC}) = \{(S,I) \mid S$ is a compact topological inverse Clifford monoid and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;

$\mathbf{Ob}(\mathscr{TB}_{\mathrm{fin}}\mathscr{SL}) = \{(S,I) \mid S$ is a topological semilattice with unity and zero and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{SL}) = \{(S,I) \mid S$ is a countably compact topological semilattice with unity and zero and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{SL}) = \{(S,I) \mid S$ is a compact topological semilattice with unity and zero and I is a non-empty finite set$\}$, and if S is a trivial semigroup then we identify (S,I) and (S,J) for all non-empty sets I and J;
$\mathbf{Ob}(\mathscr{TB}_{\mathrm{fin}}\mathscr{S}_2) = \{(S,I) \mid S$ is a topological monoid with two idempotent zero and unity and I is a non-empty finite set$\}$;
$\mathbf{Ob}(\mathscr{TB}_{\mathrm{fin}}\mathscr{G}) = \{(S,I) \mid S$ is a topological group and I is a non-empty finite set$\}$;
$\mathbf{Ob}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{G}) = \{(S,I) \mid S$ is a countably compact topological group and I is a non-empty finite set$\}$;
$\mathbf{Ob}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{G}) = \{(S,I) \mid S$ is a compact topological group and I is a non-empty finite set$\}$;

(ii) $\mathbf{Mor}(\mathscr{TB}_{\mathrm{fin}}\mathscr{I})$, $\mathbf{Mor}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{I})$, $\mathbf{Mor}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{I})$, $\mathbf{Mor}(\mathscr{TB}_{\mathrm{fin}}\mathscr{IC})$, $\mathbf{Mor}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{IC})$, $\mathbf{Mor}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{IC})$, $\mathbf{Mor}(\mathscr{TB}_{\mathrm{fin}}\mathscr{SL})$, $\mathbf{Mor}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{SL})$, $\mathbf{Mor}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{SL})$,
$\mathbf{Mor}(\mathscr{TB}_{\mathrm{fin}}\mathscr{S}_2)$, $\mathbf{Mor}(\mathscr{TB}_{\mathrm{fin}}\mathscr{G})$, $\mathbf{Mor}(\mathscr{CCTB}_{\mathrm{fin}}\mathscr{G})$, and $\mathbf{Mor}(\mathscr{CTB}_{\mathrm{fin}}\mathscr{G})$ consist of corresponding triples $(h,u,\varphi)\colon (S,I)\to(S',I')$, which satisfy condition (5).

Obviously, these categories are subcategories of the category $\mathscr{TB}_{\mathrm{fin}}$. The second series of categories is defined as follows:

(i) Let $\mathbf{Ob}(\mathscr{B}^*_{\mathrm{fin}}(\mathscr{TIS}))$ be all finite topological Brandt λ^0-extensions of topological inverse monoids S with zeros in the class of topological inverse semigroups such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CCTIS}))$ be all countably compact topological Brandt λ^0-extensions of topological inverse monoids S with zeros in the class of topological inverse semigroups such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CTIS}))$ be all compact topological Brandt λ^0-extensions of topological inverse monoids S with zeros such that S has $\mathcal{B}^*$-property and every idempotent of S lies in the center of S;
Let $\mathbf{Ob}(\mathscr{B}^*_{\mathrm{fin}}(\mathscr{TICS}))$ be all finite topological Brandt λ^0-extensions of topological inverse Clifford monoids with zeros in the class of topological inverse semigroups;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CCTICS}))$ be all countably compact topological Brandt λ^0-extensions of topological inverse Clifford monoids with zeros in the class of topological inverse semigroups;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CTICS}))$ be all compact topological Brandt λ^0-extensions of topological inverse Clifford monoids with zeros;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{TSL}))$ be all finite topological Brandt λ^0-extensions of topological semilattices with zeros and identities;

Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CCTSL}))$ be all countably compact topological Brandt λ^0-extensions of topological semilattices with zeros and identities in the class of topological inverse semigroups;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CTSL}))$ be all compact topological Brandt λ^0-extensions of topological semilattices with zeros and identities;
Let $\mathbf{Ob}(\mathscr{B}^*_{\rm fin}(\mathscr{TS}_2))$ be all finite topological Brandt λ^0-extensions of topological monoids with two idempotents zeros and identities;
Let $\mathbf{Ob}(\mathscr{B}^*_{\rm fin}(\mathscr{TG}))$ be all topological inverse Brandt semigroups with finite bands;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CCTG}))$ be all 0-simple countably compact topological inverse semigroups;
Let $\mathbf{Ob}(\mathscr{B}^*(\mathscr{CTG}))$ be all 0-simple compact topological inverse semigroups;

(ii) Let $\mathbf{Mor}(\mathscr{B}^*_{\rm fin}(\mathscr{TIS}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{CCTIS}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{CTIS}))$, $\mathbf{Mor}(\mathscr{B}^*_{\rm fin}(\mathscr{TICS}))$,

$\mathbf{Mor}(\mathscr{B}^*(\mathscr{CCTICS}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{CTICS}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{TSL}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{CCTSL}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{CTSL}))$, and $\mathbf{Mor}(\mathscr{B}^*_{\rm fin}(\mathscr{TS}_2))$ be continuous homomorphisms of corresponding topological Brandt λ^0-extensions of corresponding topological semigroups and let $\mathbf{Mor}(\mathscr{B}^*_{\rm fin}(\mathscr{TG}))$, $\mathbf{Mor}(\mathscr{B}^*(\mathscr{CCTG}))$, and $\mathbf{Mor}(\mathscr{B}^*(\mathscr{CTG}))$ be non-trivial continuous homomorphisms of the corresponding topological inverse Brandt semigroups.

Theorem 5.12 implies:

Proposition 5.13. *$\boldsymbol{B}$ is a full representative functor from $\mathscr{TB}_{\rm fin}\mathscr{I}$ [resp., $\mathscr{CCTB}_{\rm fin}\mathscr{I}$, $\mathscr{CTB}_{\rm fin}\mathscr{I}$, $\mathscr{TB}_{\rm fin}\mathscr{IC}$, $\mathscr{CCTB}_{\rm fin}\mathscr{IC}$, $\mathscr{CTB}_{\rm fin}\mathscr{IC}$, $\mathscr{TB}_{\rm fin}\mathscr{SL}$, $\mathscr{CCTB}_{\rm fin}\mathscr{SL}$, $\mathscr{CTB}_{\rm fin}\mathscr{SL}$, $\mathscr{TB}_{\rm fin}\mathscr{S}_2$, and $\mathscr{TB}_{\rm fin}\mathscr{G}$] into $\mathscr{B}^*_{\rm fin}(\mathscr{TIS})$ [resp., $\mathscr{B}^*(\mathscr{CCTIS})$, $\mathscr{B}^*(\mathscr{CTIS})$, $\mathscr{B}^*_{\rm fin}(\mathscr{TICS})$, $\mathscr{B}^*(\mathscr{CCTICS})$, $\mathscr{B}^*(\mathscr{CTICS})$, $\mathscr{B}^*(\mathscr{TSL})$, $\mathscr{B}^*(\mathscr{CCTSL})$, $\mathscr{B}^*(\mathscr{CTSL})$, $\mathscr{B}^*_{\rm fin}(\mathscr{TS}_2)$, and $\mathscr{B}^*_{\rm fin}(\mathscr{TG})$].*

Therefore Propositions 4.3 and 5.13 imply:

Corollary 5.14. *The categories $\mathscr{TB}_{\rm fin}\mathscr{SL}$, [resp., $\mathscr{CCTB}_{\rm fin}\mathscr{SL}$, $\mathscr{CTB}_{\rm fin}\mathscr{SL}$], and $\mathscr{B}^*(\mathscr{TSL})$ [resp., $\mathscr{B}^*(\mathscr{CCTSL})$, $\mathscr{B}^*(\mathscr{CTSL})$] are isomorphic.*

Gutik and Repovš [13] proved that every countably compact 0-simple topological inverse semigroup S is isomorphic to the topological Brandt λ^0-extension $\left(B^0_\lambda(G),\tau_B\right)$ of a topological group G in the class of topological inverse semigroups for some finite cardinal λ. This implies the following:

Proposition 5.15. *$\boldsymbol{B}$ is a full representative functor from $\mathscr{CCTB}_{\rm fin}\mathscr{G}$ [resp., $\mathscr{CTB}_{\rm fin}\mathscr{G}$] into $\mathscr{B}^*(\mathscr{CCTG})$ [resp., $\mathscr{B}^*(\mathscr{CTG})$].*

Comfort and Ross [5] proved that the Stone-Čech compactification of a pseudocompact topological group is a topological group. Therefore the functor of the Stone-Čech compactification β from the category of pseudocompact topological groups back into itself determines a monad. Similarly, since the Stone-Čech compactification of a countably compact 0-simple topological inverse semigroup is a compact 0-simple topological inverse semigroup [13, Theorem 3], we get the following:

Corollary 5.16. *The functor of the Stone-Čech compactification $\beta\colon \mathscr{B}^*(\mathscr{CCTG}) \to \mathscr{B}^*(\mathscr{CCTG})$ determines a monad.*

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References

[1] I. Bucur and A. Deleanu, *Introduction to the Theory of Categories and Functors*, John Willey and Sons, Ltd., London, New York and Sidney, 1968.
[2] J. H. Carruth, J. A. Hildebrant, and R. J. Koch, *The Theory of Topological Semigroups*, Vols I and II, Marcell Dekker, Inc., New York and Basel, 1983 and 1986.
[3] A. H. Clifford, *Matrix representations of completely simple semigroups*, Amer. J. Math. **64** (1942), 327–342.
[4] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vols. I and II, Amer. Math. Soc. Surveys **7**, Providence, R.I., 1961 and 1967.
[5] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacif. J. Math. **16** (1966), 483–496.
[6] C. Eberhart and J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc. **144** (1969), 115–126.
[7] R. Engelking, *General Topology*, 2nd Ed., Heldermann, Lemgo, 1989.
[8] L. M. Gluskin, *Simple semigroups with zero*, Doklady Akademii Nauk SSSR **103**:1 (1955), 5–8 (in Russian).
[9] O. V. Gutik, *On Howie semigroup*, Mat. Metody Phis.-Mech. Polya. **42**:4 (1999), 127–132 (in Ukrainian).
[10] O. V. Gutik and K. P. Pavlyk, *H-closed topological semigroup and Brandt λ-extensions*, Mat. Metody Phis.-Mech. Polya. **44**:3 (2001), 20–28 (in Ukrainian).
[11] O. V. Gutik and K. P. Pavlyk, *On topological semigroups of matrix units*, Semigroup Forum **71**:3 (2005), 389–400.
[12] O. V. Gutik and K. P. Pavlyk, *On Brandt λ^0-extensions of semigroups with zero*, Mat. Metody Phis.-Mech. Polya. **49**:3 (2006), 26–40.
[13] O. Gutik and D. Repovš, *On countably compact 0-simple topological inverse semigroups*, Semigroup Forum **75**:2 (2007), 464–469.
[14] W. D. Munn, *Matrix representations of semigroups*, Proc. Cambridge Phil. Soc. **53** (1957), 5–12.
[15] M. Petrich, *Inverse Semigroups*, John Wiley & Sons, New York, 1984.

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